

When do we need the saltation matrix?

In this document we will see when the variational equation fails to describe the perturbations in a generic orbit. The same analysis can be carried out in closed orbits like limit cycles but we will concentrate on the more general case of a simply orbit.

Then we know that if we have the orbit $\phi(t, t_0, x_0)$ and we add a perturbation on the initial condition ($\phi(t, t_0, x_0 + \delta)$) the perturbation vectors will be given from:

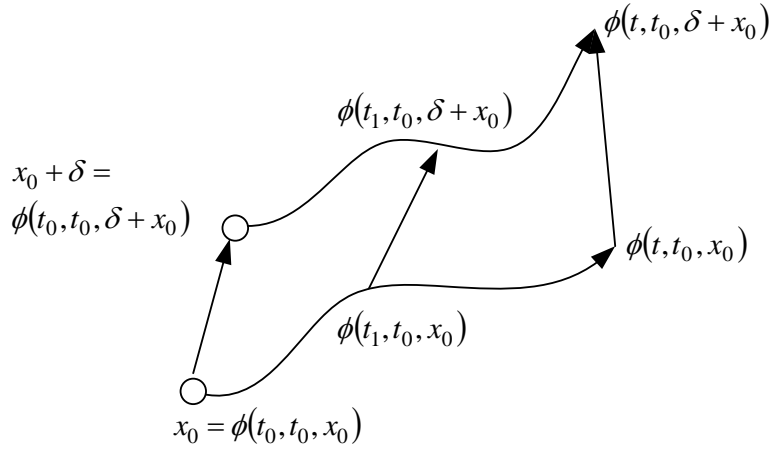


Figure 1

$$\Delta\phi(t, t_0, x_0) = \frac{\partial\phi(t, t_0, x_0)}{\partial x_0} \Delta\phi(t_0, t_0, x_0) \Leftrightarrow \quad (1)$$

$$\Delta\phi(t, t_0, x_0) = \Phi(t, t_0, x_0) \Delta\phi(t_0, t_0, x_0)$$

and to calculate the STM:

$$\frac{d}{dt} \frac{\partial\phi(t, t_0, x_0)}{\partial x_0} = \left. \frac{\partial f(x, t)}{\partial x} \right|_{x=\phi(t, t_0, x_0)} \frac{\partial\phi(t, t_0, x_0)}{\partial x_0} \quad (2)$$

Of course in that case we must know the solution $\phi(t, t_0, x_0)$.

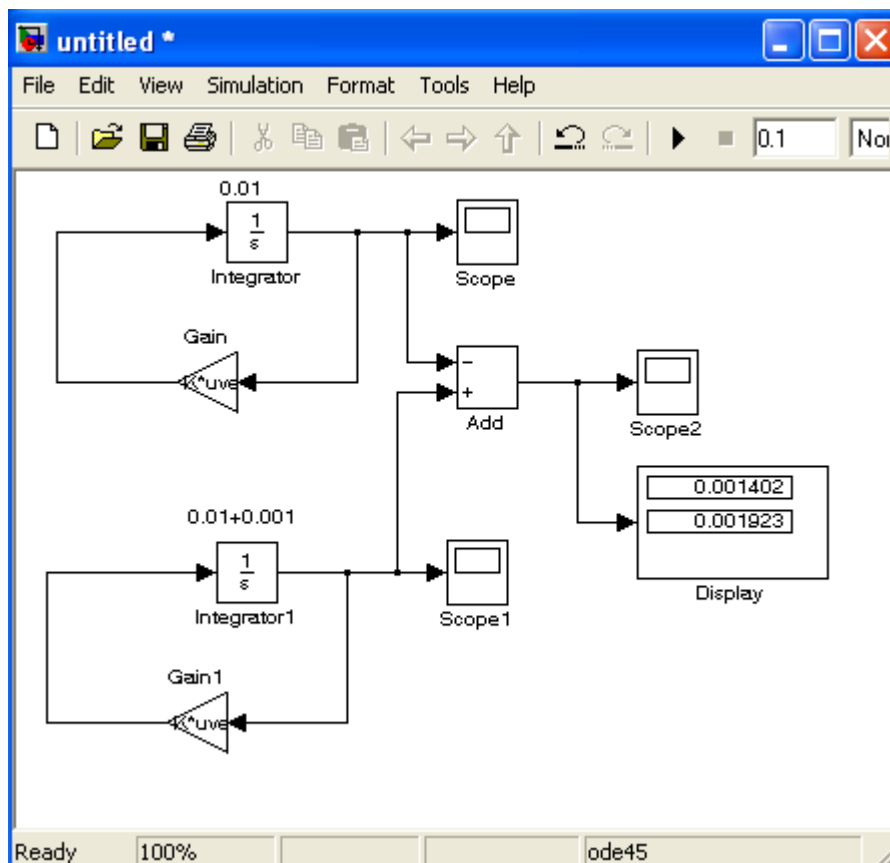
But if we concentrate on simple LTI systems ($\dot{x} = Ax$):

$$\frac{\partial\phi(t, t_0, x_0)}{\partial x_0} = e^{A(t-t_0)} \quad (3)$$

And in that case we can simplify the notation of the STM: $\Phi(t, t_0, x_0) = \Phi(t, t_0)$.

Example 1:

- Assume a simple system: $\dot{x} = f_-(x) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x$
- $x(0) = [0.01; 0.01]$.
- Initial perturbation: $\Delta x_0 = [0.001; 0.001]$.
- We expect the perturbation to be $\Delta x = e^{A \cdot 0.1} \Delta x_0$



```
>> A=[1 2;3 4]
A =
     1     2
     3     4
>> expm(A*0.1)*[0.001;0.001]
ans =
 0.00140244429646
 0.00192314579154
```

Now, let's break the evolution to 2 different paths:

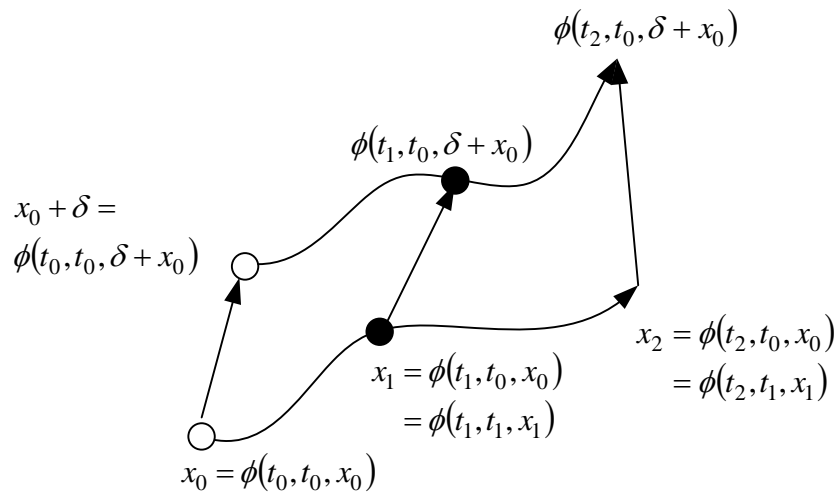


Figure 2

Analysing the above diagram:

$$\Delta\phi(t_2, t_0, x_0) = \Phi(t_2, t_0)\Delta\phi(t_2, t_0, x_0) \quad (4)$$

$$\text{Also, } \Delta\phi(t_2, t_0, x_0) = \Delta\phi(t_2, t_1, x_1) = \Phi(t_2, t_1)\Delta\phi(t_1, t_1, x_1) \quad (5)$$

$$\Delta\phi(t_1, t_1, x_1) = \Delta\phi(t_1, t_0, x_0) = \Phi(t_1, t_0)\Delta\phi(t_0, t_0, x_0) \quad (6)$$

$$\text{Thus } \Delta\phi(t_2, t_0, x_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)\Delta\phi(t_0, t_0, x_0) \quad (7)$$

In the above example this is obvious as:

$$\Delta\phi(0.1, 0, x_0) = e^{A0.1}\Delta\phi(0, 0, x_0) = e^{A0.05}e^{A0.05}\Delta\phi(0, 0, x_0)$$

But what will happen if at t_1 we have a switching to another system?

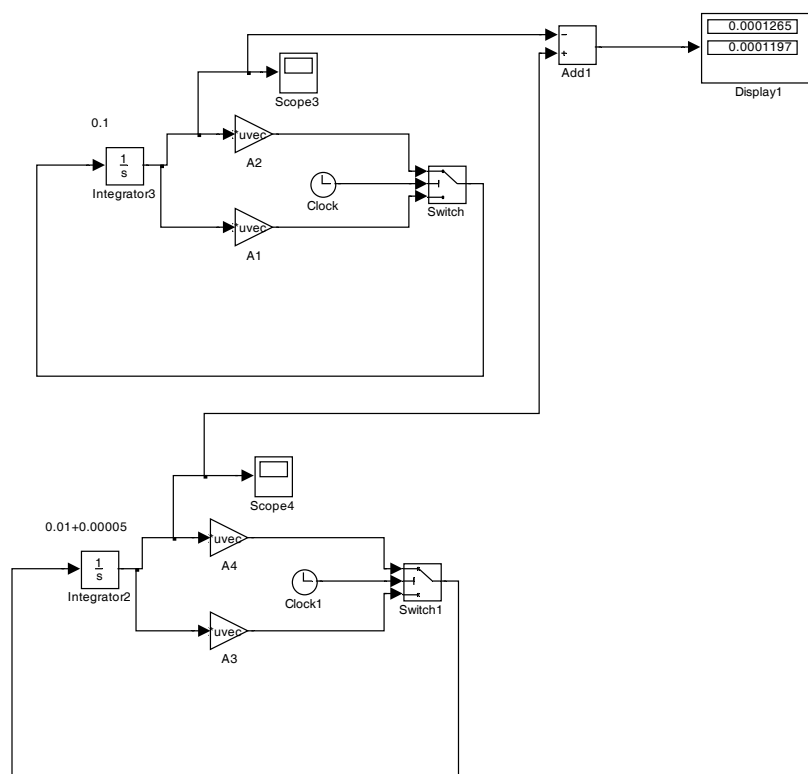
Assume that there is a switching when $t=0.05$

Assume another system $\dot{x} = f_+(x) = \begin{bmatrix} 8 & 7 \\ 6 & 5 \end{bmatrix} x$ after the switching until $t=0.1$. This

implies that the ODE is: $\dot{x} = \begin{cases} f_-(x), & t < 0.05 \\ f_+(x), & t \geq 0.05 \end{cases}$

Example 2 (The initial perturbation is 0.00005)

```
>> expm(A2*0.05)*expm(A1*0.05)*[ 0.00005; 0.00005]
ans =
1.0e-003 *
0.12648211417363
0.11974583476380
```



Thus we see that in this case $\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)$ as before. It is important to notice here that at $t=t_1$ both orbits (the original and the perturbed) switch from $f_-(x)$ to $f_+(x)$.

Now, let's assume that there is a switching surface $h(x) = x_1 - a$ but we do not change

the vector field is: $\dot{x} = \begin{cases} f_-(x), & x < a \\ f_+(x), & x \geq a \end{cases}$

Obviously this case is like Example 1.

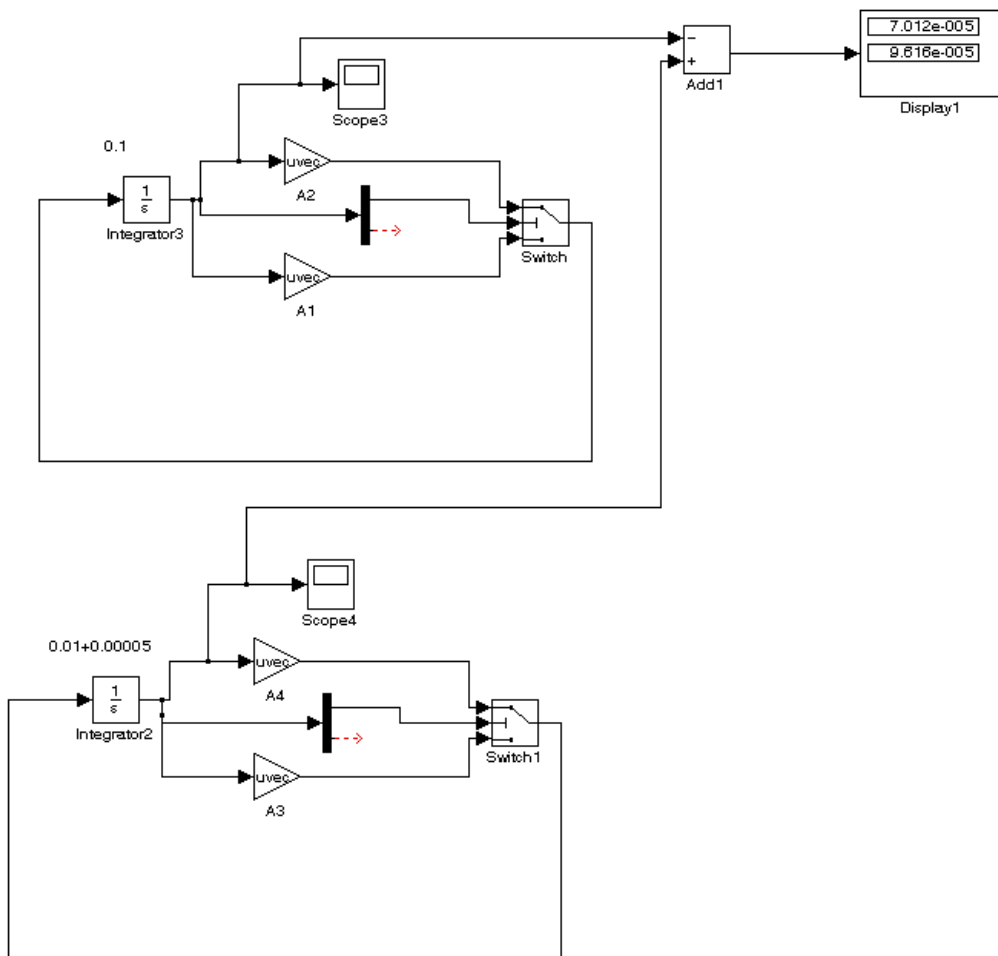
Example 3:

```
>> expm(A1*0.05)*expm(A1*0.05)*[ 0.00005; 0.00005]

ans =

1.0e-004 *

0.70122214823013
0.96157289577083
```



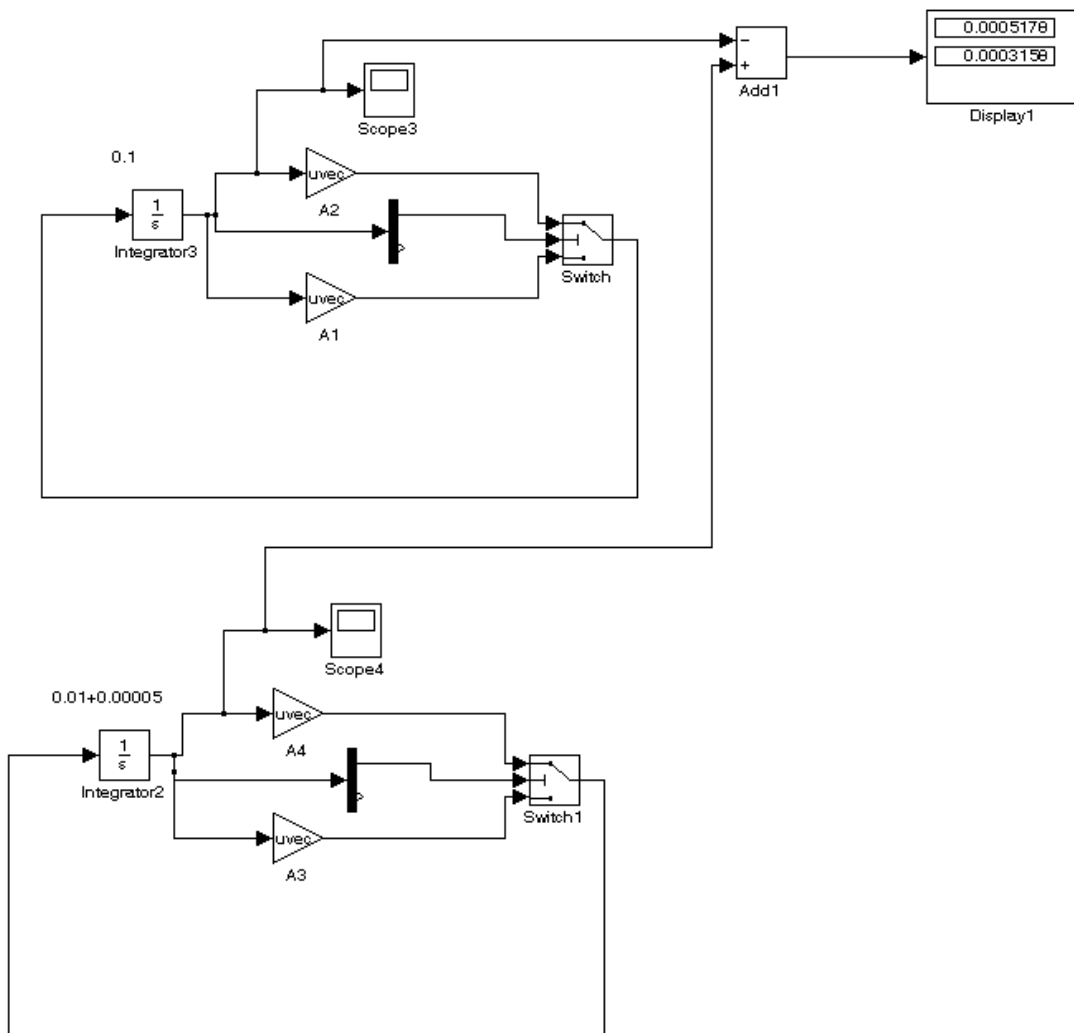
As a final example let's see the more generic (and the only interesting) case where we

$$\text{have a switching in the vector field: } \dot{x} = \begin{cases} f_-(x), & x < a \\ f_+(x), & x \geq a \end{cases}$$

Example 4:

```
>> A2=[8 7;6 5]
>> A1=[1 2;3 4]
>> expm(A2*0.05)*expm(A1*0.05)*[0.0001;0.0001]/2
ans =
1.0e-003 *
0.12648211417363
0.11974583476380
```

The numerical solution is:



Obviously there is a problem which is due to the fact that the original and perturbed orbit, do not cross the switching manifold at the same time. Thus we have to use the saltation (or correction) matrix.

Then $h_x(x) = [1 \ 0]^T$ and $h_t(x) = 0$

The saltation matrix is: $S = I + \frac{(f_+ - f_-)h_x^T}{h_x^T f_- + h_t}$

Thus $(f_+ - f_-)h_x^T = \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} [1 \ 0] = \begin{bmatrix} 7x_1 + 5x_2 & 0 \\ 3x_1 + x_2 & 0 \end{bmatrix}$

```
>> x=expm(A*0.05)*[0.01;0.01];
x1=x(1); x2=x(2);
S=eye(2)+((A2-A1)*[x1;x2]*[1 0])/([1 0]*A1*[x1;x2]);
expm(A2*0.05)*S*expm(A1*0.05)*[0.0001;0.0001]/2
ans =
  1.0e-003 *
  0.51424633736069
  0.31317261374928
```

Thus we see that in that case we need the saltation matrix (see the file Discontinuity Maps).

We have to be careful because if the perturbation is big then the linear approximations needed to derive the saltation matrix will no longer hold. For that reason I chose very small perturbations (0.00005) and the simulation time is only 0.05s.